Mesures, variabilité et incertitudes

Exercice 3 : Mesure de g avec un pendule simple

As a first example, suppose that we measure g, the acceleration of gravity, using a simple pendulum. The period of such a pendulum is well known to be $T = 2\pi\sqrt{l/g}$, where l is the length of the pendulum. Thus, if l and T are measured, we can find g as

$$g = 4\pi^2 l/T^2. \tag{3.28}$$

This result gives g as the product or quotient of three factors, $4\pi^2$, l, and T^2. If the various uncertainties are independent and random, the fractional uncertainty in our answer is just the quadratic sum of the fractional uncertainties in these factors. The factor $4\pi^2$ has no uncertainty, and the fractional uncertainty in T^2 is twice that in T:

$$\frac{\delta(T^2)}{T^2} = 2\frac{\delta T}{T}.$$

Thus, the fractional uncertainty in our answer for g will be

$$\frac{\delta g}{g} = \sqrt{\left(\frac{\delta l}{l}\right)^2 + \left(2\frac{\delta T}{T}\right)^2}. \tag{3.29}$$

Suppose we measure the period T for one value of the length l and get the results[5]

$$l = 92.95 \pm 0.1 \text{ cm},$$
$$T = 1.936 \pm 0.004 \text{ s}.$$

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Exercice 4 : Accélération d'un chariot sur une pente

$$g_{\text{best}} = \frac{4\pi^2 \times (92.95 \text{ cm})}{(1.936 \text{ s})^2} = 979 \text{ cm/s}^2.$$

To find our uncertainty in g using (3.29), we need the fractional uncertainties in l and T. These are easily calculated (in the head) as

$$\frac{\delta l}{l} = 0.1\% \quad \text{and} \quad \frac{\delta T}{T} = 0.2\%.$$

Substituting into (3.29), we find

$$\frac{\delta g}{g} = \sqrt{(0.1)^2 + (2 \times 0.2)^2}\,\% = 0.4\%;$$

from which

$$\delta g = 0.004 \times 979 \text{ cm/s}^2 = 4 \text{ cm/s}^2.$$

Thus, based on these measurements, our final answer is

$$g = 979 \pm 4 \text{ cm/s}^2.$$

Having found the measured value of g and its uncertainty, we would naturally compare these values with the accepted value of g. If the latter has its usual value of 981 cm/s^2, the present value is entirely satisfactory.

If this experiment is repeated (as most such experiments should be) with different values of the parameters, the uncertainty calculations usually do not need to be repeated in complete detail. We can often easily convince ourselves that all uncertainties (in the answers for g) are close enough that no further calculations are needed; sometimes the uncertainty in a few representative values of g can be calculated and the remainder estimated by inspection. In any case, the best procedure is almost always to record the various values of l, T, and g and the corresponding uncertainties in a single table. (See Problem 3.40.)

Finally, according to (3.33), the required acceleration is the product of (3.35) and (3.36). Multiplying these equations together (and combining the fractional uncertainties in quadrature), we obtain

$$a = (0.125 \text{ cm} \pm 2\%) \times (698 \text{ s}^{-2} \pm 9\%)$$
$$= 87.3 \text{ cm/s}^2 \pm 9\%$$

or

$$a = 87 \pm 8 \text{ cm/s}^2. \tag{3.37}$$

This answer could now be compared with the expected acceleration $g\sin\theta$, if the latter had been calculated.

When the calculations leading to (3.37) are studied carefully, several interesting features emerge. First, the 2% uncertainty in the factor $l^2/2s$ is completely swamped by the 9% uncertainty in $(1/t_2^2) - (1/t_1^2)$. If further calculations are needed for subsequent trials, the uncertainties in l and s can therefore be ignored (so long as a quick check shows they are still just as unimportant).

Another important feature of our calculation is the way in which the 2% and 3% uncertainties in t_1 and t_2 grow when we evaluate $1/t_1^2$, $1/t_2^2$, and the difference $(1/t_2^2) - (1/t_1^2)$, so that the final uncertainty is 9%. This growth results partly from taking squares and partly from taking the difference of large numbers. We could imagine extending the experiment to check the constancy of a by giving the cart an initial push, so that the speeds v_1 and v_2 are both larger. If we did, the times t_1 and t_2 would get smaller, and the effects just described would get worse (see Problem 3.42).

Exercice 5 : Incertitude sur une grandeur physique fonction de deux autres grandeurs

the steps just outlined. The uncertainty in q due to δx alone, which we denote by δq_x, is given by (3.49) as

$$\begin{aligned}\delta q_x &= \text{(error in } q \text{ due to } \delta x \text{ alone)} \\ &= \left|\frac{\partial q}{\partial x}\right| \delta x \qquad (3.50) \\ &= |2xy - y^2|\delta x = |12 - 4| \times 0.1 = 0.8.\end{aligned}$$

Similarly, the uncertainty in q due to δy is

$$\begin{aligned}\delta q_y &= \text{(error in } q \text{ due to } \delta y \text{ alone)} \\ &= \left|\frac{\partial q}{\partial y}\right| \delta y \qquad (3.51) \\ &= |x^2 - 2xy|\delta y = |9 - 12| \times 0.1 = 0.3.\end{aligned}$$

Finally, according to (3.47), the total uncertainty in q is the quadratic sum of these two partial uncertainties:

$$\begin{aligned}\delta q &= \sqrt{(\delta q_x)^2 + (\delta q_y)^2} \qquad (3.52) \\ &= \sqrt{(0.8)^2 + (0.3)^2} = 0.9.\end{aligned}$$

Thus, the final answer for q is

$$q = 6.0 \pm 0.9.$$

Dans les exercices 3, 4 et 5, l'incertitude-type est noté $\delta(T)$ et non pas $u(T)$ comme dans le cours et le programme.

Exercice 6 : Estimation d'une constante de raideur

Il suffit ici de faire le calcul de la moyenne : $\bar{k} = \frac{1}{N}\sum_{i=1}^{N} k_i$ et de l'incertitude type : $u(k) = \frac{\sigma_k}{\sqrt{N}}$ avec $\sigma_k = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left(k_i - \bar{k}\right)^2}$. On trouve : $k = 13{,}16 \pm 0{,}06$ N.m^{-1}.